Statistical Reversion Toward the Mean: More Universal Than Regression Toward the Mean

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Schmittlein discussed the lack of universality of regression toward the mean. The present note emphasizes the universality of a similar effect, dubbed "reversion" toward the mean, defined as the shift in conditional expectation of the upper or lower portion of a distribution. Reversion toward the mean is a useful concept for statistical reasoning in applications and is more self-evidently plausible than regression toward the mean.

KEY WORDS: Probability mixture models; Regression to the mean; Reversion to the mean.

1. INTRODUCTION

In a recent commentary, Schmittlein (1989) demonstrated that the phenomenon of statistical regression toward the mean is by no means universal, even in mixture models where universality might have been hoped for. The purpose of the present note is to point out that a closely related phenomenon is much more nearly universal and to present a heuristic proof of this universality that is easily accessible to nonstatisticians.

2. REGRESSION AND REVERSION TOWARD THE MEAN

Let X_1 and X_2 be random variables with joint distribution function F. Assume that X_1 and X_2 have the same marginal distribution and let μ denote their common mean. The distribution F exhibits regression toward the mean if, for all $c > \mu$,

$$\mu \leq E[X_2 \mid X_1 = c] < c, \tag{1}$$

with the reverse inequalities holding for $c < \mu$.

As noted by Schmittlein (1989), Galton (1877) originally used the term *reversion* rather than *regression*. Let us resurrect this archaic term for a new use, and say that F exhibits *reversion toward the mean* if, for any c,

$$\mu \leq E[X_2 \mid X_1 > c] < E[X_1 \mid X_1 > c] \tag{2a}$$

and

$$\mu \geq E[X_2 \mid X_1 < c] > E[X_1 \mid X_1 < c]. \tag{2b}$$

(To avoid trivialities, we restrict attention throughout to values of c for which the conditional expectations are defined.) Clearly, regression toward the mean implies reversion toward the mean, but not vice versa.

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As defined by (2), reversion toward the mean occurs when the conditional mean of the upper or lower portion of the distribution shifts, or reverts, toward the unconditional mean μ. In many applications the upper or lower portion of interest would be a *small* portion (a tail) of the distribution, but reversion is not restricted to this case; note that (2) places no restriction on the location of c relative to μ.

Reversion can serve as well as regression to motivate statistical cautionary tales. For example, educational researchers can be warned that a group of school children selected because their performance is below some cutoff would be expected, on the average, to show improvement when observed later. Or medical investigators can be alerted that patients selected for levels of serum potassium higher than some cutoff would be expected, on the average, to show reduced levels when observed later.

The phenomenon (2) has been discussed in this kind of context (for example, by Davis 1976, 1986; McDonald, Mazzuca, and McCabe 1983), but terminology is often vague and the distinction between (1) and (2) is frequently blurred. Recently Senn (1990) has emphasized that (2) is a phenomenon of practical importance and has suggested that both (1) and (2) should be considered forms of regression toward the mean. It might be less confusing, however, to give distinct names to these distinct phenomena. Since the conditional expectation function $f(x) = E[X_2 \mid X_1 = x]$ is generally called the "regression" function (Rao 1973, p. 264; Dixon and Massey 1983, p. 210–211; Kendall, Stuart, and Ord 1987, p. 524), it seems appropriate to continue to refer to (1) as "regression" and to choose a different name for (2).

3. THE UNIVERSALITY OF REVERSION TOWARD THE MEAN

To investigate the universality of reversion toward the mean, it is helpful to split the definition into two parts by noting that (2) holds iff

$$E[X_2 \mid X_1 > c] < E[X_1 \mid X_1 > c], \quad \text{for all } c, \tag{3a}$$

$$E[X_2 \mid X_1 < c] > E[X_1 \mid X_1 < c], \quad \text{for all } c, \tag{3b}$$

and

$$E[X_2 \mid X_1 > c] \geq \mu, \qquad E[X_2 \mid X_1 < c] \leq \mu, \quad \text{for all } c. \tag{4}$$

The condition (3) can be called *reversion to the mean or beyond;* (3a) asserts that the mean of the upper portion of the distribution reverts to a lower position, and (3b) asserts contrariwise for the mean of the lower portion. The additional requirement (4) assures that the reversion cannot go beyond the mean μ.

Reversion to the mean or beyond is virtually a universal phenomenon. Only the nondegeneracy condition

$$\Pr[X_2 > c \mid X_1 > c] < 1, \qquad \Pr[X_2 < c \mid X_1 < c] < 1, \quad \text{for all } c \tag{5}$$

is required to assure that the inequalities in (3) are strict.

Proposition. If X_1 and X_2 are identically distributed and (5) holds, then (3) holds; that is, F exhibits reversion to the mean or beyond.

Here are two proofs of the proposition. The first is a straightforward mathematical proof. (A different mathematical proof, assuming X_1 and X_2 nonnegative, was given by McDonald et al. 1983.)

Mathematical Proof. Let I_i, for $i = 1, 2$, denote indicator random variables defined by

$$\begin{aligned} I_i &= 1 \quad \text{if } X_i > c \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and let $J = I_1 - I_2$. Then

$$E[X_2 I_1] = E[X_2 I_2] + E[X_2 J] = E[X_1 I_1] + E[X_2 J]. \tag{6}$$

(The second equality follows because X_1 and X_2 are identically distributed.) The fact that $X_2 < c$ when $J = 1$ and $X_2 > c$ when $J = -1$, together with (5), implies that

$$\begin{aligned} E[X_2 J] &< c \Pr[J = 1] - c \Pr[J = -1] \\ &= c\, E[J] \\ &= 0, \end{aligned}$$

so that (6) yields

$$E[X_2 I_1] < E[X_1 I_1],$$

from which (3a) follows immediately. Relation (3b) is proved similarly.

The second proof of the Proposition is an easily understood heuristic argument, framed in terms of IQ, a variate that has the same distribution at any age.

The Red T-Shirt Argument. Let X_1 and X_2 represent IQ at age 8 and at age 18, respectively. Visualize a very long row of chairs, each occupied by an 8-year-old child; each chair bears a label with the child's IQ, and the chairs are arrayed in nondecreasing order of IQ from left to right. Suppose a kindly teacher decides to reward the "smart" children whose IQ exceeds c; say, $c = 120$. She places a marker on the rightmost chair labeled 120, and all children sitting to the right of the marker receive red T-shirts. What happens to the mean IQ of the red T-shirted children as they grow from age 8 to 18? Imagine that the chairs and labels remain in place (representing the stationarity of the distribution) while the children get up, have various adventures (but retain their red T-shirts), and return at age 18 to take a chair corresponding to their current IQ. Possibly all red T-shirts are still sitting to the right of the marked chair; in this case [(5) being violated] the mean IQ of the red T-shirt group has not changed. But, if *any* red T-shirts have moved to the left of the marked chair, then clearly the mean IQ of the red T-shirt group must have decreased. This proves the first relation in (3); the second would be argued similarly.

If, in addition to having identical marginals, the distribution F satisfies the condition (4), then F exhibits reversion toward the mean (*not* beyond). Condition (4) asserts a weak form of positive dependence between X_1 and X_2, weaker than positive quadrant dependence as defined by Lehmann (1966), but stronger than nonnegative correlation. Reversion toward the mean is universal, then, among distributions with X_1 and X_2 identically distributed and weakly positively dependent in the sense of (4).

In longitudinal studies, where X_1 and X_2 are measurements on the same subject, it would usually be reasonable to assume that (4) should hold. In such a setting, (4) simply asserts that the group of subjects whose initial scores are higher [lower] than c will score higher [lower] than average when measured subsequently; this will certainly be true if the expected future score of an individual is a monotonically increasing function of his initial score.

As mathematical models for longitudinal studies, Schmittlein (1989) considered latent trait mixture models, in which X_1 is an observation from a mixture distribution of the form $\int G(x \mid \theta)\, dH(\theta)$ and X_2 is another observation for the same value of θ. It is easy to show that in these models (4) will hold if the distribution functions $G(x \mid \theta)$ are monotone in θ. Thus, for example, any distribution generated by mixing Poissons will exhibit reversion toward the mean, even though, as in examples 4.1 and 4.2 of Schmittlein (1989), it may not exhibit regression toward the mean. Similarly, any distribution generated by mixing normals (with equal variances) on their mean will exhibit reversion toward the mean, even though it may not exhibit regression toward the mean, and, indeed, it will generally (if it is unimodal) exhibit regression toward the *mode,* as shown by Das and Mulder (1983). As further examples, any distribution generated by mixing binomials (with equal n's) on their success probability, or by mixing gammas (with equal scale parameters) on their shape parameter, or by mixing gammas (with equal shape parameters) on their scale parameter, will exhibit reversion toward the mean.

The preceding discussion has assumed X_1 and X_2 to be identically distributed. If they are not, the notions of regression and reversion toward the mean can be generalized to mean that the standardized random variables $X_i^* = (X_i - \mu_i)/\sigma_i$ exhibit regression or reversion toward zero, where μ_i and σ_i are the mean and standard deviation of X_i, for $i = 1, 2$. Clearly, reversion toward the mean in this generalized sense is universal whenever X_1 and X_2 belong to the same location-scale family and are positively dependent in the sense that the X_i^* satisfy (4)

4. THE MAGNITUDE OF THE REVERSION EFFECT

When both regression and reversion toward the mean occur, we can compare the magnitudes of the two effects. The comparison is very simple in the case of linear conditional expectations, that is, when $E[X_2 \mid X_1] = \alpha + \beta X_1$, which, because X_1 and X_2 are identically distributed, can be expressed as

$$E[X_2 \mid X_1] = (1 - \rho)\mu + \rho X_1, \tag{7}$$

where ρ is the correlation between X_1 and X_2. As noted by Schmittlein (1989), if (7) holds, then regression toward the mean always occurs. Assuming (7), the relative magnitude of the regression shift is

$$\frac{c - E[X_2 \mid X_1 = c]}{c - \mu} = 1 - \rho,$$

and the relative magnitude of the reversion shift is

$$\frac{E[X_1 \mid X_1 > c] - E[X_2 \mid X_1 > c]}{E[X_1 \mid X_1 > c] - \mu}$$

$$= \frac{E[X_1 \mid X_1 > c] - \{(1 - \rho)\mu + \rho E[X_1 \mid X_1 > c]\}}{E[X_1 \mid X_1 > c] - \mu}$$

$$= 1 - \rho.$$

Thus in the case of linear conditional expectations, the relative magnitudes of the reversion shift and the regression shift are equal. [Incidentally, for the IQ example of Section 3, ρ is roughly .7, as reported by Anastasi (1988), p. 337].

5. CONCLUSION

Reversion toward the mean—the shift of the conditional expectation of the upper or lower portion of a distribution—is a phenomenon that shares much of the intuitive content of the more familiar regression toward the mean. In contrast with regression toward the mean, reversion toward the mean occurs under very general conditions, requiring essentially only that the variates under discussion be identically distributed and (in a weak sense) positively dependent. Moreover, a vivid heuristic argument is available that not only can convince nonstatisticians that the phenomenon occurs but also can convey an intuitive understanding of why its occurrence is inevitable.

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